ACTION PRINCIPLE AND LAGRANGIAN WITH HIGHER ORDER DERIVATIVES

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(*Received, July* 16, 1959)

ABSTRACT. We have obtained here certain commutation relations of field quantities (field operators and their derivatives) for space-like separation of the points when the Lagrangian density contains derivatives of field operators of order higher than first by using Schwinger's operator principle of Stationary Action. The commutators thus obtained are quite complicated, and the consistency of this procedure can only be discussed in individual cases

I. INTRODUCTION

In a series of interesting papers, Schwinger has considered quantum field theory in a very general way (Schwinger (1951), (1953) and (1954)). The starting point of this series was a further generalisation of the Action Principle by using the ideas of quantum mechanics. However, all these considerations were confined to the case of a Lagrangian containing only the first order derivative of the field operators. We shall try to generalise this procedure further when the Lagrangian contains derivatives of the field operators of any finite order. We note that this gives rise to a lot of complications even for a single field without spin, and we are able to only write down a number of commutation relations for space-like separation of the field quantities The consistency of the many such relations that we deduce is not possible to prove; but they are necessary conclusions from the Action Principle and the fundamental ideas of quantum mechanics. We shall have occasion to demonstrate this method in greater detail for the case when the Lagrangian contains only second order derivatives of the field operators and gives rise to a particularly simple type of field equations proposed by Bhabha (1950) and Thirring (1950).

We shall first very briefly outline the Action Principle that we are going to utilise.

Let us consider two space-like surfaces σ_1 and σ_2 on which the dynamical variables are known On σ_1 and on σ_2, we construct complete set of commuting observables ζ_1 and ζ_2 respectively which have corresponding eigenvalues ζ_1' and ζ_2'. Thus, we can write the state vectors on the surfaces σ_1 and σ_2 in terms of these eigenvalues as $|\zeta_1, \sigma_1>$ and $|\zeta_2', \sigma_2>$. The transformation function here is.

$$<\zeta_1, \sigma_1 | \zeta_2, \sigma_2>$$

A variation of this transformation function is given as

$$\delta < \zeta'_1, \sigma_1 | \zeta_2, \sigma_2 > = i < \zeta_1, \sigma_1 | \delta W_{12} | \zeta_2, \sigma_2 > \quad (1)$$

also

$$\delta W_{13} + \delta W_{32} = \delta W_{12}, \quad \dots \quad (2)$$

$$\delta W_{13} = -\delta W_{31} \quad \dots \quad (3)$$

and

$$\delta W_{12}^{\dagger} = \delta W_{12} \quad \dots \quad (4)$$

Since the state vectors on any surface form a complete orthonormal set, when we normalise these state vectors to unity we obtain,

$$\sum_{\zeta'} | \zeta', \sigma > < \zeta', \sigma | = 1. \quad \dots \quad (5)$$

Consistent with the properties (2) to (4), we assume that there exists a hermitian function L of the field operators such that

$$W_{12} = \int_{\sigma_2}^{\sigma_1} L|a|d^4x$$

where

$$\delta W_{12} = \delta(W_{12}).$$

The above L is the Lagrangian density and will give us the complete information about the dynamical system The operator sets ζ_1 and ζ_2 are similarly constructed from the field variables on the two surfaces σ_1 and σ_2 snd in the neighbourhood of these surfaces.

We can get a variation of W_{12} in equation (6) by varying the field operators and their derivatives on the surfaces σ_1 and σ_2 and in the region interior to them, and also by varying the surfaces σ_1 and σ_2 themselves. Now, as a result of such variation, let the new and the old state-vectors be related by the unitary transformation

$$|(\zeta+\delta_0\zeta)', \sigma + \sigma\delta > = \bigcup | \zeta', \sigma >$$

Writing $\bigcup = \exp(-iF) \simeq 1-iF$, where $F = F(\zeta, \delta_0\zeta; \sigma, \delta\sigma)$ is hermitian and the generator of the infinitesimal transformation on the surface σ due to the above type of variations, we obtain.

$$\delta | \zeta', \sigma > = -iF(\zeta, \delta_0\zeta; \sigma, \delta\sigma) | \zeta', \sigma >$$

which, in conjunction with equation (1) gives us,

$$\delta W_{12} = F(\zeta_1, \delta_0\zeta_1; \sigma_1, \delta\sigma_1) - F(\zeta_2, \delta_0\zeta_2; \sigma_2, \delta\sigma_2) \quad \dots \quad (7)$$

Now, in postulate (6), we have, (for definiteness, with σ_1 later than σ_2)

$$\begin{aligned}\delta W_{12} &= \delta \int_{\sigma_2}^{\sigma_1} L(x) d^4x \\ &= \int_{\sigma_2}^{\sigma_1} \delta L d^4x + \int_{\sigma_2}^{\sigma_1+\delta\sigma_1} L d^4x + \int_{\sigma_2+\delta\sigma_2}^{\sigma_2} L d^4x \\ &= \int_{\sigma_2}^{\sigma_1} \delta L d^4x + \left(\int_{\sigma_1} - \int_{\sigma_2} \right) L \delta x_\mu \, d\sigma_\mu \qquad \ldots \quad (8)\end{aligned}$$

where x_μ lying either on σ_1 or σ_2 changes to $x_\mu + \delta x_\mu$ due to the variation of the surfaces, and $d\sigma_\mu = n_\mu d\sigma$ with $d\sigma$ denoting an element of the surface σ and n_μ being the normal to the surface in the time-increasing direction.

2. ACTION PRINCIPLE WHEN THE LAGRANGIAN DERIVATIVES OF ORDER HIGHER THAN THE FIRST

We now proceed to apply the Action Principle to the case when the Lagrangian contains derivatives of the field operators of order higher than the first. We consider for simplicity $L(x)$ to be a function of a single field operator $\phi(x)$ and its derivatives of any finite order. This is not a loss of generality in obtaining the field equations or the commutation rules, as will be clear from what follows. We shall further also restrict our considerations to the case of a scalar or a pseudo-scalar field such that they remain unchanged under rotations, and the complications due to the transformations of field operators with spin are eliminated. Thus we write,

$$L = L(\phi, \partial_\mu \phi, \partial_\mu \partial_\nu \phi, \ldots).$$

$$\left(\partial_\mu = \frac{\partial}{\partial x_\mu} \right)$$

Then we obtain δL due to a variation $\delta_0 \phi$ of the field operator ϕ as

$$\delta L = \frac{\partial L}{\partial \phi} \delta_0 \phi + \frac{\partial L}{\partial(\partial_\mu \phi)} \delta_0(\partial_\mu \phi) + \frac{\partial L}{\partial(\partial_\mu \partial_\nu \phi)} \delta_0(\partial_\mu \partial_\nu \phi) + \ldots \qquad \ldots \quad (9)$$

We now use the fact that $\delta_0(\partial_\mu \partial_\nu \ldots \phi) = \partial_\mu \partial_\nu \ldots (\delta_0 \phi)$ and find, with

$$\partial_\mu \partial_\nu \partial_\lambda \ldots (n \text{ times}) = \partial^{(n)}{}_{\mu\nu\lambda} \ldots \qquad \ldots \quad (10)$$

an n-th order partial differential operator, that, for any function $f_{\mu\nu\lambda} \ldots (x)$ *symmetric in all its indices*,

$$\begin{aligned} f_{\mu\nu\lambda} \ldots (x)\partial^{(n)}{}_{\mu\nu\lambda} \ldots (\delta_0\phi) &= (-1)^n\, \partial^{(n)}{}_{\mu\nu\lambda}\ldots f_{\mu\nu\lambda} \ldots (x)\, \delta_0\phi \\ &+\partial_\mu[f_{\mu\nu\lambda} \ldots (x)\partial^{(n-1)}{}_{\nu\lambda} \ldots (\delta_0\phi) \\ &-\partial_\nu f_{\mu\nu\lambda} \ldots (x)\partial^{(n-2)}{}_{\lambda} \ldots (\delta_0\phi) \\ &+\partial^{(2)}{}_{\nu\lambda}\, f_{\mu\nu\lambda} \ldots (x)\partial^{(n-3)} \ldots (\delta_0\phi) \\ &- \ldots +(-1)^{(n-1)}\, \partial^{(n-1)}{}_{\nu\lambda} \ldots f_{\mu\nu\lambda} \ldots (x)(\delta_0\phi)] \end{aligned} \quad \ldots \quad (11)$$

Hence, putting

$$f_{\mu\nu\lambda} \ldots (x) = \frac{\partial L}{\partial(\partial_\mu\partial_\nu\partial_\lambda \ldots \phi)},$$

we obtain from relations (9) and (11), on rearranging the terms on the right hand side such that the coefficients of $\delta_0\phi$, $\partial_\mu(\delta_0\phi)$, ... are conveniently separated in the term with ∂_μ,

$$\begin{aligned} \delta L = \left[\frac{\partial L}{\partial\phi} - \partial_\mu \frac{\partial L}{\partial(\partial_\mu\phi)} + \partial_\mu\partial_\nu \frac{\partial L}{\partial(\partial_\mu\partial_\nu\phi)} - \right] \delta_0\phi \\ +\partial_\mu[\pi_\mu(\delta_0\phi) + \pi_{\mu\nu}\partial_\nu(\delta_0\phi) + \pi_{\mu\nu\lambda}\, \partial_\nu\partial_\lambda(\delta_0\phi) + \ldots], \end{aligned} \quad \ldots \quad (12)$$

where

$$\pi_\mu = \frac{\partial L}{\partial(\partial_\mu\phi)} - \partial_\nu \frac{\partial L}{\partial(\partial_\mu\partial_\nu\phi)} + \partial_\nu\partial_\lambda \frac{\partial L}{\partial(\partial_\mu\partial_\nu\partial_\lambda\phi)} - \ldots,$$

$$\pi_{\mu\nu} = \frac{\partial L}{\partial(\partial_\mu\partial_\nu\phi)} - \partial_\lambda \frac{\partial L}{\partial(\partial_\mu\partial_\nu\partial_\lambda\phi)} + \ldots \quad .. \quad (13)$$

Applying Gauss theorem to the term with ∂_μ in the expression (12), we now obtain from equation (8),

$$\delta W_{12} = \int^{\sigma_1} \left[\frac{\partial L}{\partial\phi} - \partial_\mu \frac{\partial L}{\partial(\partial_\mu\phi)} + \partial_\mu\partial_\nu \frac{\partial L}{\partial(\partial_\mu\partial_\nu\phi)} - \ldots\right] (\delta_0\phi)d^4x + F(\sigma_1) - F(\sigma_2) \quad (14)$$

where

$$F(\sigma) = \int_\sigma [\pi_\mu(\delta_0\phi) + \pi_{\mu\nu}\partial_\nu(\delta_0\phi) + \pi_{\mu\nu\lambda}\partial_\nu\partial_\lambda(\delta_0\phi) + \ldots + \delta x_\mu]d\sigma_\mu \quad \ldots \quad (15)$$

Comparing expression (14) with (7), the operator principle of Stationary Action asserts that W_{12} must be stationary for the variation of the field operators in the interior of the volume bounded by σ_1 and σ_2. This gives rise to the field equation for the operator ϕ as

$$\frac{\partial L}{\partial \phi}-\partial_\mu \frac{\partial L}{\partial(\partial_\mu\phi)}+\partial_\mu\partial_\nu\frac{\partial L}{\partial(\partial_\mu\partial_\nu\phi)}-\ldots=0 \qquad \ldots \quad (16)$$

When the field equations are satisfied, we obtain for the variation of W_{12}

$$\delta W_{12}=F(\sigma_1)-F(\sigma_2) \qquad \ldots \quad (17)$$

with the definition of $F(\sigma)$ given by equation (15). We note that the principle of Stationary Action gives us the same form (7) of $\delta\, W_{12}$ as demanded by quantum mechanics

We shall now consider equation (15) in detail, since this is the equation that gives the possible commutation relations and also the energy-momentum tensor. For this purpose we need to write the right hand side integral of equation (15) in terms of the *independent* variations of different quantities on the surface σ However, we note that once the value of $\delta_0\phi$ on the surface is determined, the variations $\partial_\mu(\delta_0\phi)$ $\partial_\mu\partial_\nu(\delta_0\phi)$,... are not completely arbitrary, but their variations only in the normal direction are arbitrary. In order to separate the arbitrary increments corresponding to the surface we first define

$$\partial^{(0)}=n_\mu\partial_\mu, \qquad \partial_{t\mu}=\partial_\mu+n_\mu\partial^{(0)} \qquad \ldots \quad (18)$$

such that

$$n_\mu\partial_{t\mu}=\partial^{(0)}+n_\mu n_\mu\partial^{(0)}=0,\ (n_\mu n_\mu=-1).$$

We may note that the normal n_μ to a space-like surface is a time-like four-vector, and is exactly defined, with $d^{(n)}x_\mu$ a displace-ment normal to the surface,

$$n_\mu=\frac{d^{(n)}x_\mu}{\sqrt{(dx_0)^2-(d\vec{x})^2}}\,.$$

In equation (18), $\partial_{t\mu}$ indicates the differential operator in the tangential direction at any point of the surface σ, and similarly, $\partial^{(0)}$ indicates the normal (time-increasing) directional derivative at any point. We shall henceforward either take the surfaces σ_1 and σ_2 as plane surfaces, or, as will be sufficient for our purpose, as surfaces such that the tangential derivatives of the normal vanishes upto a sufficiently large but finite order. Let us further introduce the notation for the partial tengential derivative of the k-th order as

$$\partial^{(t,k)}{}_{\nu\lambda}\ldots{}_{(k\ \text{indices}}\equiv\partial_{t\nu}\ \partial_{t\lambda}\ \ldots\ (k\ \text{times}) \qquad \ldots \quad (19)$$

in addition to notation (10). Then, if the Lagrangian density contains upto the the N-th order partial derivatives of the field operator, we assume that

$$\partial^{(t,k)}{}_{\nu\lambda}\ldots(n_\mu)=0 \qquad \text{for}\quad 1\leqslant k\leqslant N-1$$

In these notations, of course, the zeroeth order partial derivative as usual is the unity operator.

To obtain commutators, we shall also further take that

$$(\partial^{(0)})^k(\delta_0\phi)$$

approaches zero sufficiently rapidly for infinite spatial distances, so that, for any $f_\nu(x)$ on σ,

$$\int_\sigma \partial_{t\nu}[f_\nu(x)(\partial^{(0)})^k(\delta_0\phi)]d\sigma = 0 \qquad \dots \quad (20)$$

whenever $$k \leqslant N-1.$$

Now, once we know $(\partial^{(0)})^k(\delta_0\phi)$ on the surface σ, $\partial_{t\nu}(\partial^{(0)k})\,(\delta_0\phi)$ is determined, and hence only $(\partial^{(0)})^{k+1}(\delta_0\phi)$ is arbitrary. Hence, we shall have to find out the coefficients of $(\partial^{(0)})^k(\delta_0\phi)$. $0 \leqslant k \leqslant N-1$, on the right hand side of equation (15), these being independent variations on the surface

Now, with $n \leqslant N-1$,

$$\int_\sigma \pi_{\mu\nu\lambda}\dots(n+1 \text{ indices})\partial_{\nu\lambda}{}^{(n)}\dots(\delta_n\phi)d\sigma_\mu$$

$$= \int_\sigma \pi_{\mu\nu\lambda}\,\overline{(n+1} \text{ indices})\,(\partial_{t\nu} - n_\nu\partial^{(0)})(\partial_{t\lambda} - n_\lambda\,\partial^{(0)})\dots \text{ to } n \text{ factors } (\delta_0\phi)d\sigma_\mu.$$

$$= \int_\sigma (-1)^n[\partial^{(t,n)}{}_{\nu\lambda}\dots\pi_{\mu\nu\lambda}\dots(x)(\delta_0\phi) + \binom{n}{1} n_\nu\ \partial^{(t,n-1)}{}_\lambda\dots\pi_{\mu\nu\lambda}\,(x)(\partial^{(0)})(\delta_0\phi) + \dots$$

$$+ \binom{n}{k} n_\nu n_\lambda\dots(k \text{ terms})\partial^{(t,n-k)}{}_\alpha\ \pi_{\mu\nu\lambda\dots\alpha\dots}(x)(\partial^{(0)})^k(\delta_0\phi)$$

$$+\dots+ \binom{n}{n}\ n_\nu n_\lambda\dots\pi_{\mu\nu\lambda}\dots(x)(\partial^{(0)})^n\ (\delta_0\phi)]d\sigma_\mu \qquad \dots \quad (21)$$

In deducing equation (21) we have utilised the symmetry of $\pi_{\mu\nu\lambda}\dots$. The consecutive terms from the last backwards of this formula have been obtained by partial integration with the help of equation (20) mentioned earlier.

Thus, in equation (15) we have, collecting coefficients of $(\partial^{(0)})^k(\delta_0\phi)$ separately,

$$F(\sigma)$$

$$= \int_\sigma \sum_{n=0}^{N-1} \pi_{\mu\nu\lambda}\dots\overline{(n}+1 \text{ indices})\ \partial^{(n)}{}_{\nu\lambda}\dots(\delta_0\phi)\ d\sigma_\mu$$

$$= \int_\sigma \sum_{n=0}^{N-1} (-1)^n \sum_{k=0}^{n}\binom{n}{k} n_\nu n_\lambda\ \dots\ (k \text{ terms})$$

$$\times\,\partial^{(t,n-k)}{}_\alpha\pi_{\mu\nu\lambda}\ \dots\alpha\dots\ (x)(\partial^{(0)})^k(\delta_0\phi)d\sigma_\mu$$

$$= \int_\sigma \sum_{k=0}^{N-1} \sum_{n=k}^{N-1} (-1)^n n_\nu n_\lambda \ldots (k \text{ terms})\ \partial_\alpha^{(t,n-k)} \pi_{\mu\nu\lambda} \ldots \alpha \ldots (x) (\partial^{(0)} (\delta_0 \phi) d\sigma_\mu$$

$$\equiv \int_\sigma \sum_{k=0}^{N-1} \pi_\mu^{[\sigma,k]} (x) (\partial^{(0)k} (\delta_0 \phi) d\sigma_\mu \qquad \ldots \quad (22)$$

where

$$\pi_\mu^{[\sigma,k]} = \sum_{n=k}^{N-1} (-1)^n \binom{n}{k} n_\nu n_\lambda \ldots (k \text{ times}) \partial_\alpha^{(t,n-k)} \pi_{\mu\nu\lambda} \ldots {}_\alpha \ldots (x) \qquad \ldots \quad (23)$$

The $\pi_\mu^{[\sigma,k]}$ are the fields conjugate to $(\partial^{(0)})^k \phi$ with $0 \leqslant k \leqslant N-1$. When $N = 1$ such that we have only $\pi^{[\sigma,0]}$, there is only one field conjugate to $\phi(x)$.

The fact that $\pi_\mu^{[\sigma,k]}(x')$ are conjugate fields can be obtained when we find the commutation relationships of these with the field operators $(\partial^{(0)})^l \phi(x)$ for space-like separation of the points x and x'. To obtain these, we require the variation $\delta_0\phi(x')$ on the surface and in its immediate neighbourhood (which determines $(\partial^{(0)})^k(\delta_0\phi) = \delta_0((\partial^{(0)})^k\phi)$ on the surface σ as arbitrary variations). Also, we take $\delta x_\mu = 0$ so that displacement of the surface is not considered. Now, let any operator G suffer a change $G \rightarrow G + \delta G$ due to this variation. This change in the operator is also associated with the change in the state-vector as we have seen previously. Thus we have,

$$\begin{aligned} \delta < \zeta', \sigma | G | \zeta'', \sigma > &= < \zeta', \sigma | \delta G | \zeta'', \sigma > \\ &= (\delta < \zeta', \sigma |) G | \zeta'', \sigma > + < \zeta', \sigma | G (\delta | \zeta'', \sigma >) \\ &= i < \zeta', \sigma | [F(\sigma), G] | \zeta'', \sigma > \end{aligned}$$

such that

$$[G,\ F(\sigma)] = i\delta G. \qquad \ldots \quad (24)$$

Hence, taking $G = (\partial^{(0)})\ \phi(x)$, $0 \leqslant l \leqslant N-1$, by equation (24) we get,

$$[(\partial^{(0)})^l \phi(x),\ F(\sigma)] = i\delta_0((\partial^{(0)})^l \phi(x)) \qquad \ldots \quad (25)$$

where how we have

$$F(\sigma) = \int_{\sigma(x)} \sum_{k=0}^{N-1} \pi_\mu^{[\sigma,k]}(x') \delta_0((\partial'^{(0)})^k \phi(x')) d\sigma_\mu(x') \qquad \ldots \quad (26)$$

where $\sigma(x)$ is a spece-like surface passing through a point x. Equations (25) and (26) are explicitly written as

$$\sum_{k=0}^{N-1} \int_{\sigma(x)} [(\partial^{(0)})^l \phi(x),\ \pi_\mu^{[\sigma,k]}(x') \partial_0((\partial'^{(0)})^k \phi(x'))] d\sigma_\mu(x') = i\delta_0((\partial^{(0)})^l \phi(x). \quad \ldots \quad (27)$$

In equation (27), we now take $\delta_0((\partial^{(0)})^l \phi(x))$ on the surface as nonzero only at very small neighbourhood of the point x, and also that $\delta_0((\partial'^{(0)})^k\phi(x')) = 0$ for $k \neq l$.

Then we obtain,

$$[(\partial^{(0)})^l\phi(x), \pi_\mu{}^{[\sigma l]}(x')\delta_0((\partial^{(0)})^k\phi(x'))]$$

$$= i\delta_\mu{}^{(\sigma)}(x-x')\delta_0((\partial^{(0)})^l\phi(x)) \quad .. \quad (28)$$

where $\delta_\mu{}^{(\sigma)}(x-x')$ is the surface δ-function such that for any regular function $f(x')$ defined on the surface,

$$\int_{\sigma(x)} f(x')\delta_\mu{}^{(\sigma)}(x-x')d\sigma_\mu(x') = f(x) \quad ... \quad (29)$$

It may be emphasized that this delta function originates since we are at liberty to take arbitrary variations $\delta_0((\partial'^{(0)})^l\phi(x'))$ as not equal to zero only at a very small neighbourhood of the point x.

Let us now take $\delta_0((\partial'^{(0)})^k\phi(x'))$ as arbitrary where $k \neq l$ and $\delta_0((\partial'^{(0)})^l\phi(x')) = 0$. Then we obtain, for $k \neq l$

$$[(\partial^{(0)})^l\phi(x), \pi^{[\sigma,k]}{}_\mu(x')\delta_0((\partial'^{(0)})^k\phi(x'))] = 0 \quad ... \quad (30)$$

Combining equations (28) and (30), we must have, for space-like separation of the points x and x',

$$[(\partial^{(0)})^l\phi(x), \pi^{[\sigma,k]}{}_\mu(x')\delta_0((\partial'^{(0)})^k\phi(x'))] = i\delta^{kl}\delta^{(\sigma)}{}_\mu(x-x')\delta_0((\partial^{(0)})^l\phi(x)), \quad .. \quad (31)$$

where δ^{kl} is the Kronecker delta.

At present we do not have any reason to say that the many commutators in equation (31) are consistent with each other However, from the Action Principle and basic ideas of quantum mechanics, the above commutators are obtained, and only those fields for which they are self-consistent should be taken.

ACKNOWLEDGMENTS

The author wishes to acknowledge his sincere thanks to Prof. D. Basu, Indian Association for the Cultivation of Science, for suggesting the problem.

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